

Math 2042 Chapter One

Eigen Value Eigen Vector.

Characteristic polynomial

Defn: Let A be square matrix $n \times n$. the characteristic polynomial of A is the determinant of the $n \times n$ matrix $A - \lambda I_n$. This is a polynomial of degree n in λ .

Defn: An eigen vector of an $n \times n$ matrix A , corresponding to eigen value $\lambda = d_i$, is a nonzero n -element column vector x_i that satisfies the matrix equation $Ax_i = d_i x_i$ or $Ax = \lambda x$.

or. the eigen vector x_i of the $n \times n$ matrix A , corresponding to the eigen value $\lambda = d_i$ is a solution of the homogeneous equation $(A - \lambda I_n)x_i = 0$

Remark! the eigen value of $n \times n$ matrix A are the n -zeros of the polynomial $p(\lambda) = \det(A - \lambda I_n)$. Equivalently, the n roots of the n th degree polynomial equation $\det(A - \lambda I) = 0$

✓ A matrix with complex coefficients will have complex eigen values.

✓ the scalar equation $\det(A - \lambda I_n) = 0$ is called the characteristic polynomial of A where A is square matrix and I_n is identity matrix.

✓ The expression $\det(A - \lambda I_n)$ is often referred to as characteristic polynomial. This will be denoted by $\psi_A(\lambda)$.

✓ The root of the characteristic equation is eigen value.

Ex- Find the characteristic polynomial, the eigen value and Eigen vector(s) of the matrix.

(1) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ (2) $A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$ (3) $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 0 & 1 & -2 \end{bmatrix}$

So, (1) $|(A - \lambda I_2)| = 0$ is the characteristic eqn of A.

$$\Rightarrow \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow 4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$$

$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$ is the characteristic polynomial of A.

$$\Rightarrow (\lambda - 3)(\lambda - 2) = 0$$

$\Rightarrow \lambda = 3$ or $\lambda = 2$ are the eigenvalues of A.

The eigenvector of A corresponding to eigen value λ is given by $(A - \lambda I_2)x = 0$, for $x \neq 0$.

Thus, $(A - \lambda I_2)x = 0$

Case 1 When $\lambda = 2$ we have

$$\begin{pmatrix} 4-2 & -1 \\ 2 & 1-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{matrix} 2x - y \\ 2x - y \end{matrix} = 0 \Rightarrow y = 2x.$$

or $x = \frac{1}{2}y$.

$$E_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x = \frac{1}{2}y, y \in \mathbb{R}, y \neq 0 \right\}$$

$$= \left\{ \begin{pmatrix} \frac{1}{2}y \\ y \end{pmatrix} : y \in \mathbb{R}, y \neq 0 \right\}$$

$= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} y : y \in \mathbb{R}, y \neq 0 \right\}$ is the eigen vector corresponding to eigen value $\lambda = 2$.

Case 2 When $\lambda = 3$ we will obtain

$$\begin{pmatrix} 4-3 & -1 \\ 2 & 1-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{matrix} x - y = 0 \\ 2x - 2y = 0 \end{matrix} \Rightarrow x = y.$$

$$\text{Thus } E_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x = y, y \in \mathbb{R}, y \neq 0 \right\}$$

$$= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} y : y \in \mathbb{R}, y \neq 0 \right\}$$

$= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} y : y \in \mathbb{R}, y \neq 0 \right\}$ is the eigen vector of A corresponding to eigen value $\lambda = 3$.

So, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are the eigen vectors of A.

#2. $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix}$. Then the characteristic polynomial is given by

$$|A - \lambda I_3| = 0.$$

So, $\begin{vmatrix} 2-\lambda & 1 & -1 \\ 3 & 2-\lambda & -3 \\ 3 & 1 & -2-\lambda \end{vmatrix} = 0$ is the characteristic polynomial of A .

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & -1 \\ 3 & 2-\lambda & -3 \\ 3 & 1 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda) \begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} - (-1) \begin{vmatrix} 3 & -3 \\ 3 & -2-\lambda \end{vmatrix} + (-1) \begin{vmatrix} 3 & 2-\lambda \\ 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(2-\lambda)(-2-\lambda)+3] - [6-3\lambda+9] + (-1)[3-(3)(2-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)[-4-2\lambda+2\lambda+\lambda^2+3] - [3-3\lambda] - (3-6+3\lambda) = 0$$

$$\Rightarrow (2-\lambda)[\lambda^2-1] - 3+3\lambda+3-3\lambda = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-1) = 0 \text{ is the characteristic polynomial of } A.$$

Thus, the eigen value is given by $\lambda = 2$ or $\lambda = \pm 1$.

The eigen vector is given by $(A - \lambda I_n)x = 0$ for $x \neq 0$.

Case 1 When $\lambda = -1$ we have

$$\begin{pmatrix} 2-(-1) & 1 & -1 \\ 3 & 2-(-1) & -3 \\ 3 & 1 & -2-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} 3x + y - z = 0 \\ 3x + 3y - 3z = 0 \\ 3x + y - z = 0 \end{cases} \Rightarrow \begin{cases} 3x + y - z = 0 \\ -2y + 2z = 0 \Rightarrow y = z \end{cases}$$

$$\text{So, } 3z + y - y = 0 \Rightarrow x = 0.$$

$$\text{Thus, } E_{-1} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y = z, y \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} y : y \in \mathbb{R}_{\neq 0} \right\} \text{ is the eigen value of } A \text{ corresponding to } \lambda = -1.$$

$$E_{-1} = v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ is the eigen vector of } A \text{ corresponding to } \lambda = -1.$$

Case 2 When $\lambda = 1$ we have

$$\begin{pmatrix} 2-1 & 1 & -1 \\ 3 & 2-1 & -3 \\ 3 & 1 & -2-1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{is the eigenvector of } A \text{ corresponding to eigen value } \lambda = 1.$$

$$\Rightarrow \begin{cases} x + y - z = 0 \\ 3x + y - 3z = 0 \\ 3x + y - 3z = 0 \end{cases} \Rightarrow \begin{cases} -2x + 2z = 0 \\ x = z \end{cases}$$

$$\Rightarrow \boxed{y = 0}.$$

$$\text{Thus, } E_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = z, z \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} z : z \in \mathbb{R}_{\neq 0} \right\} \quad \text{is the eigenvector of } A \text{ corresponding to eigen value } \lambda = 1.$$

Case 3 When $\lambda = 2$ we have

$$\begin{pmatrix} 0 & 1 & -1 \\ 3 & 0 & -3 \\ 3 & 1 & -4 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{cases} y - z = 0 \\ 3x - 3z = 0 \\ 3x + y - 4z = 0 \end{cases} \Rightarrow \begin{cases} y = z \\ \text{and } x = z = y \end{cases}$$

$$\Rightarrow \boxed{3z + y - 4z = 0}$$

$$E_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = y = z, z \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} z : z \in \mathbb{R}_{\neq 0} \right\} \quad \text{is the eigenvector of } A \text{ corresponding to eigen value } \lambda = 2.$$

#2. Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Then β generates $\mathbb{R}^3$

Let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$

$$\Rightarrow \begin{cases} a + b + c = 0 \\ a + c = 0 \\ a + b + c = 0 \end{cases} \Rightarrow \begin{cases} b = -c \\ a = -c \end{cases} \Rightarrow a = b = -c = 0.$$

Thus, β is linearly independent.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = b + c \\ y = a + c \\ z = a + b + c \end{cases} \Rightarrow \begin{cases} x - b = c \\ y - a = c \end{cases}$$

$$z = y - c + b + x - b = y + x - c$$

$$\Rightarrow z - y - x = -c \Rightarrow \boxed{c = z + y - x}$$

$$X - b = c$$

$$\Rightarrow X - b = X + y - z$$

$$\Rightarrow -b = X + y - z - X$$

$$\Rightarrow -b = y - z \Rightarrow \boxed{b = z - y}$$

$$y - c = a \Rightarrow y - (X + y - z) = a$$

$$\Rightarrow y - X - y + z = a$$

$$\Rightarrow \boxed{z - X = a}$$

$$\text{Thus, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (z - x) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (z - y) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (x + y - z) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus, β generates $V = \mathbb{R}^3$.

Hence $\beta = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

And hence, spectra of A is a basis of $\mathbb{R}^3$.

Remark! (1) If A is an $n \times n$ matrix. Then a non-zero vector v in $\mathbb{R}^n$ is said to be an eigen vector of A if and only if $Av = \lambda v$ for some scalar λ .

(The scalar with this property is called an eigen value of A and the vector v is called an eigen vector corresponding to A .)

(2) The set of all distinct eigen values of a matrix A is called the spectrum of A and denoted by $\sigma(A)$.

(3) If a matrix A is order n , then an n -dimensional non-null vector v is called an eigen vector of A if there exists an eigen value λ such that $Av = \lambda v$.

(4) The set of all distinct eigen vectors of A is called eigen space.

(5) The procedure to solve $Av = \lambda v$

(1) Compute the $\det A - \lambda I_n$. This $\det$ is polynomial of degree n . It starts with $(-\lambda)^n$

(2) Find the root of the polynomial

— The n^{th} roots are the eigen values of A .

(3) Find the eigen vector solve the equation $AX = \lambda X$.

Theorem prove the following

- (1) if $\det A = 0$ then $0 \in \delta(A)$ and conversely.
- (2) if $\lambda \in \delta(A)$, $\lambda \neq 0$ and A is non-singular matrix, then $\lambda^{-1} \in \delta(A^{-1})$.
- (3) $\delta(A) = \delta(P^{-1}AP)$, where P is non-singular matrix.
- (4) if A is (real) symmetric, then $\delta(A) \subseteq \mathbb{R}$.
- (5) if A is real skew-symmetric then $\delta(A) \subseteq i\mathbb{R}$.
- (6) if A is real orthogonal matrix and $\lambda \in \delta(A)$ then $|\lambda| = 1$.
- (7) if $\lambda \in \delta(A)$ and A is orthogonal, then $\lambda^{-1} \in \delta(A)$.

Proof: (1) if $\det A = 0$, then A is singular matrix and its characteristic equation of A is given by

$$\psi_A(\lambda) = (-1)^n \lambda^n + (\text{trace}) \lambda^{n-1} + \dots + (\text{the sum of principal minors of order } n-1) \lambda.$$

$\Rightarrow$ Evidently 0 is a root of $\psi_A(\lambda)$.

Conversely, if 0 is a root of characteristic polynomial of $\psi_A(\lambda)$, then $\psi_A(0) = 0$. or $\det A = 0$.

(2) Assume $\lambda \in \delta(A)$, $\lambda \neq 0$ and A is non-singular.

Since $\lambda \in \delta(A)$, $\det(A - \lambda I) = 0$ WTS λ^{-1} is eigen value of A^{-1} or A^{-1} satisfies the equation $\det(A^{-1} - \lambda^{-1} I) = 0$.

Note that $(\det A^{-1} - \lambda^{-1} I) = \det(\lambda A^{-1} - I_n)$

$$= \frac{\lambda^n}{\lambda^n} \det(\lambda A^{-1} - I) \frac{\det A}{\det A}$$

$$= \frac{\det(\lambda A^{-1}A - I_n A)}{\lambda^n \det A}$$

$$= \frac{\det(I_n - A)}{\lambda^n \det A} = 0$$

$$\Rightarrow - \frac{\det(A - \lambda I)}{\lambda^n \det A} = 0 \quad (\because \lambda \in \delta(A))$$

Thus, λ^{-1} satisfies the equation $\det(A^{-1} - \lambda^{-1} I) = 0$.
as $\det A \neq 0$, $\lambda \neq 0$.

(3) $\delta(A) = \delta(P^{-1}AP)$ where P is non-singular matrix, ~~then $\lambda^{-1} \in \delta(A^{-1})$~~
 Assume that P is a non-singular matrix and $\lambda \in \delta(A)$.

WTS $\delta(A) = \delta(P^{-1}AP)$

$$\begin{aligned} \text{We see that } |P^{-1}AP - \lambda I_n| &= |\bar{P}^{-1}AP - \lambda \bar{P}^{-1}P| \quad (\because I = \bar{P}^{-1}P) \\ &= |\bar{P}^{-1}AP - \bar{P}^{-1}\lambda P| \quad (\because \lambda \text{ is scalar}) \\ &= |\bar{P}^{-1}(A - \lambda I_n)P| \\ &= |\bar{P}^{-1}| |A - \lambda I_n| |P| \quad (\because \det(ABC) = \det A \det B \det C) \\ &= \frac{1}{|P|} |A - \lambda I_n| |P| = \frac{|P|}{|P|} |A - \lambda I_n| \\ &= |I_n| |A - \lambda I_n| \\ &= |A - \lambda I_n| \end{aligned}$$

and hence $|A - \lambda I_n| = |\bar{P}^{-1}AP - \lambda I_n|$.

Thus, $|\bar{P}^{-1}AP - \lambda I_n| = 0$ if and only if $|A - \lambda I_n| = 0$. i.e. $\lambda(A) = \delta(P^{-1}AP)$.

(4) Assume A is real symmetric WTS $\delta(A) \subseteq \mathbb{R}$.

Since A is real symmetric, then $\bar{A}^t = A$. Also, since λ is an eigen value of A , where there exist a non-zero vector x s.t.

$$(A - \lambda I)x = 0 \text{ or } Ax = \lambda x.$$

Now taking the transpose of the conjugate both sides, we get

$$(\bar{A}x)^t = (\bar{\lambda}x)^t \text{ or } x^t \bar{A}^t = \bar{\lambda}x^t$$

$$\Rightarrow x^t \bar{A}^t = \bar{\lambda}x^t \quad (\because \bar{A}^t = A, \lambda \text{ is scalar})$$

Now, by post-multiplying by x , we get

$$x^t \bar{A}^t x = \bar{\lambda} x^t x \Leftrightarrow x^t \lambda x = \bar{\lambda} x^t x \quad (\because Ax = \lambda x)$$

$$\Leftrightarrow x^t x \lambda = \bar{\lambda} x^t x \quad (\because \lambda \text{ is scalar})$$

$$\Leftrightarrow x^t x (\lambda - \bar{\lambda}) = 0$$

$$\Leftrightarrow \lambda - \bar{\lambda} = 0 \quad (\because x^t x \neq 0)$$

$$\Leftrightarrow \lambda = \bar{\lambda}.$$

(5) therefore, $\lambda = \bar{\lambda}$, since $x^t x \neq 0$ as x is a non-zero vector.

Hence, λ is a real number. i.e. $\delta(A) \subseteq \mathbb{R}$.

#5. Assume that A is a real skew-symmetric matrix.

WTS $\delta(A) \subseteq i\mathbb{R}$.

Since A is skew matrix $A^t = -A$.

Again since λ is eigen value of A , there exists a non-zero vector x such that $(A - \lambda I_n)x = 0$ or $Ax = \lambda x$.

Now by taking transpose of the conjugate both side we get

$$(\bar{A}x)^t = (\bar{\lambda}x)^t \Rightarrow x^t \bar{A}^t = \bar{\lambda}x^t$$

by post-multiplying both side by x we get

$$-x^t A x = \bar{\lambda} x^t x \quad (\because A^t = -A)$$

$$\Leftrightarrow -x^t \lambda x = \bar{\lambda} x^t x \quad (\because Ax = \lambda x)$$

$$\Leftrightarrow -x^t x \lambda = \bar{\lambda} x^t x \quad (\because \lambda \text{ is scalar})$$

$$\Leftrightarrow (\bar{\lambda} + \lambda) x^t x = 0$$

$$\Leftrightarrow \bar{\lambda} + \lambda = 0 \text{ since } x^t x \neq 0.$$

$$\Leftrightarrow \bar{\lambda} = -\lambda.$$

Therefore, $\bar{\lambda} = -\lambda$ or λ is 0 or purely imaginary. i.e. $\delta(A) \subseteq i\mathbb{R}$.

#6. Assume that A is real orthogonal matrix and $\lambda \in \delta(A)$.

WTS $|\lambda| = 1$.

Since A is real orthogonal $\bar{A}^t A = I$ and λ is eigen value of A .

Then $\det(A - \lambda I_n) = 0$.

So, there exists a non-zero x such that $(A - \lambda I)x = 0$ or $Ax = \lambda x$.

Now taking transpose of the conjugate, we have

$$(\bar{A}x)^t = (\bar{\lambda}x)^t \text{ or } x^t \bar{A}^t = \bar{\lambda}x^t$$

post-multiplying both side by x we obtain

$$x^t \bar{A}^t A x = \bar{\lambda} x^t A x$$

$$\Leftrightarrow x^t I x = \bar{\lambda} x^t \lambda x \quad (\because A \text{ is orthogonal matrix and } \lambda \text{ is eigen value of } A \text{ then } Ax = \lambda x)$$

$$\Leftrightarrow x^t x = \bar{\lambda} \lambda x^t x$$

$$\Leftrightarrow (1 - \bar{\lambda} \lambda) x^t x = 0$$

$$\Leftrightarrow 1 - \bar{\lambda} \lambda = 0 \quad (\because x^t x \neq 0)$$

Hence, $\bar{\lambda} \lambda = 1$ since $x^t x \neq 0$ this implies $|\lambda| = 1$.

#7. Assume that $\lambda \in \delta(A)$ and A is orthogonal.

WTS $\lambda^{-1} \in \delta(A)$.

Since $\lambda \in \delta(A)$, $\det(A - \lambda I) = 0$ and $\lambda \neq 0$ w/c implies $\det A \neq 0$.
and $\det(A^t - \lambda I) = 0$. Now

$$\begin{aligned}\det(A - \lambda^{-1} I) &= \det\left(A - \frac{1}{\lambda} I\right) \\&= \frac{\det(\lambda A - A A^t)}{\lambda^n} \quad (\because A A^t = I) \text{ and } \det(kA) = k^n \det A. \\&= -\frac{\det(A) [\det(A^t - \lambda I)]}{\lambda^n} \\&= -\det(A) \cdot 0 = 0. \quad (\because \det(A - \lambda I) = 0) \\&\quad \text{and the eigen value of } A \text{ \& } A^t \text{ are equal or the same.}\end{aligned}$$

So, λ^{-1} is the eigen value of A .

i.e. $\lambda^{-1} \in \delta(A)$. Hence, the result!

Theorem The characteristic roots of diagonal matrix are elements of the leading diagonals.

Proof EX.

Thm If u is an eigen ~~value~~ Vector of the matrix A , then u cannot correspond to more than one eigen value of A .

Proof Let u be the eigen vector of the matrix A corresponding to two eigen values λ_1 & λ_2 .

$$Au = \lambda_1 u = \lambda_2 u$$

$$\Rightarrow \lambda_1 u = \lambda_2 u$$

$$\Rightarrow \lambda_1 u - \lambda_2 u = 0$$

$$\Rightarrow (\lambda_1 - \lambda_2)u = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0 \quad (\because u \text{ is eigen vector of } A \text{ corresponding to } \lambda_1 \text{ \& } \lambda_2. \text{ So, } \lambda_1 \neq \lambda_2)$$

$$\Rightarrow \lambda_1 = \lambda_2.$$

Thus u cannot be correspond to more than one eigen value.

Theorem. The set of eigen vectors of a matrix A corresponding to eigen value λ form a subspace of a vector space of the eigen vector of A .

Proof: Let λ be an eigen value of the square matrix A and $K_\lambda = \{u \in V : Au = \lambda u\}$. V being a vector space formed by eigen vectors of A ,

Let $u, v \in K_\lambda$, then $Au = \lambda u$ and $Av = \lambda v$. if $\alpha, \beta \in F$, then

$$\begin{aligned} A(\alpha u + \beta v) &= A(\alpha u) + A(\beta v) \\ &= \alpha Au + \beta Av \\ &= \alpha \lambda u + \beta \lambda v \\ &= \lambda(\alpha u + \beta v). \end{aligned}$$

$\therefore$ This shows that for $\alpha, \beta \in F$ and $u, v \in K_\lambda$, $\alpha u + \beta v \in K_\lambda$.

Hence, K_λ is a subspace of V .

Theorem The eigen value of matrix and its transpose are the same.

Proof: Let A be square matrix and A^T be its transpose.

If λ be an eigen value of A , then λ is a root of the equation.

$|A - \lambda I_n| = 0$, As the value of a determinant does not change if its rows and columns are interchanged, so we have

$$|A - \lambda I_n|^T = 0 \Rightarrow |A^T - \lambda I_n^T| = 0$$

$\Rightarrow$ This shows that λ is also the eigen value of the matrix A^T .

Theorem If $A = [a_{ij}]_{n \times n}$ is triangular matrix, then $\delta(A) = \{a_{11}, \dots, a_{nn}\}$.

Proof Ex.

Similarity of Matrices

Defn: If A and B are square matrices, then we say that B is similar to A if there exists an invertible matrix P such that $B = P^{-1}AP$.

Eg. Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Soln. Step 1 find the characteristic polynomial of A and eigen value.

$$\begin{aligned} \text{So, } \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} &= 0 \Rightarrow (1-\lambda)(4-\lambda) + 2 = 0 \\ &\Rightarrow 4 - \lambda - 4\lambda + \lambda^2 + 2 = 0 \\ &\Rightarrow \lambda^2 - 5\lambda + 6 = 0 \\ &\Rightarrow (\lambda - 2)(\lambda - 3) = 0 \end{aligned}$$

$\Rightarrow \lambda = 2$ and $\lambda = 3$ are the eigen values of A .

Step 2 find the eigenvector corresponding to eigen values of A .

Case 1. When $\lambda = 2$ we have

$$\begin{bmatrix} 1-2 & 1 \\ -2 & 4-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{aligned} -x + y &= 0 \\ -2x + 2y &= 0 \end{aligned} \Rightarrow \boxed{x = y}$$

$$\text{Thus } E_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = y, y \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} y : y \in \mathbb{R}_{\neq 0} \right\} \text{ is the eigenvector of } A \text{ corresponding to } \lambda = 2.$$

Case 2 when $\lambda = 3$ we have

$$\begin{bmatrix} 1-3 & 1 \\ -2 & 4-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{aligned} -2x + y &= 0 \\ -2x + y &= 0 \end{aligned} \Rightarrow \boxed{x = \frac{y}{2}} \text{ or } \boxed{y = 2x}$$

$$\text{Thus } E_3 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x = y, x \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} x : x \in \mathbb{R}_{\neq 0} \right\} \text{ is the eigenvector of } A \text{ corresponding to eigen value } \lambda = 3.$$

and hence $V = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is the eigenvectors of A .

Step₃: put $p = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and find the inverse of p if possible.

Here $|p| = 2 - 1 = 1 \neq 0$ Thus, p is invertible.

$$C_{11} = 2, \quad C_{12} = -1, \quad C_{21} = -1, \quad C_{22} = 1$$

$$\text{Cofactor matrix is } \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$p^{-1} = \frac{1}{\det p} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}}$$

$$\begin{aligned} \text{Step}_4: \text{ find } p^{-1} A p &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = B. \end{aligned}$$

and hence B is similar to A .

Exercise. show that $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 2 & 1 & -1 \\ 10 & 4 & -7 \\ 6 & 2 & -3 \end{bmatrix}$ or not.

Remark! (1) if B is similar to A then A is similar to B as $p^{-1} A p = B$ implies $A = (p^{-1})^{-1} B p^{-1}$ and p^{-1} is non-singular when p is non-singular.

(2) if A and B are similar then they have the same set of Eigen values.

(3) Two matrices having the same set of Eigen values may not be similar.

Eg Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ have the same Eigen values 1.

But the matrices are not similar since the 2nd matrix cannot be obtained from A as $B = p^{-1} A p$. A being identity matrix.

Note $p^{-1} A p = I$ always if $A = I$.

(4) Similarity is an equivalence relation on the set of square matrices.

i.e. it satisfies (i) $A \sim A$ b/c $A = I^{-1} A I$

$\Rightarrow$ similarity is reflexive.

(ii) $A \sim B$ then $B \sim A$. (similarity is symmetric)

(i.e. $A = S B S^{-1} \Rightarrow S^{-1} A S = S^{-1} (S B S^{-1}) S = I B I = B$.)

(iii) if $A \sim B$ and $B \sim C$ then $A \sim C$.

(i.e. $A = S B S^{-1}$ and $B = T C T^{-1}$

$$\Rightarrow A = S B S^{-1} = S (T C T^{-1}) S^{-1} \\ = S T C T^{-1} S^{-1} \\ = S T C (S T)^{-1}$$

$\Rightarrow$ Thus $A \sim C$ and it is transitive.

Theorem Similar matrices have the same

(i) determinant

(ii) characteristic polynomial

(iii) Eigen Values.

Proof Exercise.

Diagonalization

Defn: A matrix M is said to be diagonalizable if there exists an invertible matrix P and diagonal matrix D such that

$$D = P^{-1} M P.$$

Eg. Show that $B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$ is diagonalizable.

Soln Step find the Eigen vector of B .

$$\text{So, } \begin{vmatrix} -7-\lambda & -6 & -12 \\ 5 & 5-\lambda & 7 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (-7-\lambda) \begin{vmatrix} 5-\lambda & 7 \\ 0 & 4-\lambda \end{vmatrix} + 6 \begin{vmatrix} 5 & 7 \\ 1 & 4-\lambda \end{vmatrix} - 12 \begin{vmatrix} 5 & 5-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (-7-\lambda) [20 - 5\lambda - 4\lambda] + 6(20 - 5\lambda - 7) - 12(0 - (5-\lambda)) = 0$$

$$\Rightarrow (-7-\lambda) [\lambda^2 + 9\lambda + 20] + 6(-5\lambda + 13) - 12(-5 + \lambda) = 0$$

$$\Rightarrow (-7\lambda^2 + 63\lambda - 140 - \lambda^3 + 9\lambda^2 - 20\lambda) - 60\lambda + 78 + 60 - 12\lambda = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 + 43\lambda - 140 - 60\lambda + 138 - 12\lambda = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 + \lambda + 2 = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda - 2 = 0 \text{ is the characteristic polynomial of } B.$$

$$\Rightarrow (\lambda - 1)(\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = 1, \lambda = -1, \lambda = 2 \text{ are the Eigen Value of } B.$$

Step₂: find the Eigen Vector of B

Case 1 when $\lambda = -1$, we have

$$\begin{pmatrix} -7-(-1) & -6 & -12 \\ 5 & 5-(-1) & 7 \\ 1 & 0 & 4-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -6 & -6 & -12 \\ 5 & 6 & 7 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} -6x - 6y - 12z = 0 \\ 5x + 6y + 7z = 0 \\ x + 5z = 0 \end{cases}$$

$$\Rightarrow \boxed{x = -5z}$$

$$x = -5z$$

$$-25z + 6y + 7z = 0$$

$$\Rightarrow -18z + 6y = 0$$

$$\Rightarrow 6y = 18z$$

$$\Rightarrow y = 3z$$

$$\text{thus } E_{-1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = -5z, y = 3z, z \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{bmatrix} -5z \\ 3z \\ z \end{bmatrix} : z \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} z : z \in \mathbb{R}_{\neq 0} \right\} \text{ is Eigen vector corresponding to eigen value } \lambda = -1.$$

Case 2 when $\lambda = 1$, we have

$$\begin{pmatrix} -8 & -6 & -12 \\ 5 & 4 & 7 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} -8x - 6y - 12z = 0 \\ 5x + 4y + 7z = 0 \\ x + 3z = 0 \end{cases}$$

$$\Rightarrow 24z - 6y - 12z = 0$$

$$\Rightarrow 12z - 6y = 0 \Rightarrow 12z = 6y$$

$$\Rightarrow \boxed{2z = y}$$

$$\Rightarrow \boxed{x = -3z}$$

$$\text{Thus, } E_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = -3z, y = 2z, z \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} z : z \in \mathbb{R}_{\neq 0} \right\} \text{ is the Eigen vector of } B \text{ corresponding to eigen value } \lambda = 1.$$

Case 3 When $\lambda = 2$ we get

$$\begin{bmatrix} -9 & -6 & -12 \\ 5 & 3 & 7 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{cases} -9x - 6y - 12z = 0 \Rightarrow 18z - 12z + -6y = 0 \\ 5x + 3y + 7z = 0 \Rightarrow \boxed{z = y} \\ x + 2z = 0 \Rightarrow \boxed{x = -2z} \end{cases}$$

$$\text{Thus, } E_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = -2z, y = z, z \in \mathbb{R}_{\neq 0} \right\}$$

$$= \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} z : z \in \mathbb{R}_{\neq 0} \right\} \text{ is the Eigen vector of } B \text{ corresponding to Eigen value } \lambda = 2.$$

Step₃: put $P = \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and find its inverse if possible.

$$\text{Here } |P| = \begin{vmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -5 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}.$$

$$= -5(2-1) + 3(3-1) - 2(3-2) = -5 + 6 - 2 = -1 \neq 0$$

Thus, P is invertible matrix.

$$\text{So, Cofactor of } C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1, C_{12} = - \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -2, C_{13} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1.$$

$$C_{21} = - \begin{vmatrix} -3 & -2 \\ 1 & 1 \end{vmatrix} = 1, C_{22} = \begin{vmatrix} -5 & -2 \\ 1 & 1 \end{vmatrix} = -3, C_{23} = - \begin{vmatrix} -5 & -3 \\ 1 & 1 \end{vmatrix} = 2.$$

$$C_{31} = \begin{vmatrix} -3 & -2 \\ 2 & 1 \end{vmatrix} = 1, C_{32} = - \begin{vmatrix} -5 & -2 \\ 3 & 1 \end{vmatrix} = -1, C_{33} = \begin{vmatrix} -5 & -3 \\ 3 & 2 \end{vmatrix} = -1$$

$$\text{Thus Cofactor of } P = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -3 & 2 \\ 1 & -1 & -1 \end{bmatrix} \text{ Thus adjoint of } P \text{ is } \begin{bmatrix} 1 & 1 & 1 \\ -2 & 3 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\text{Thus } P^{-1} = \frac{1}{\det P} [\text{adj } P] \Rightarrow P^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 3 & 1 \\ -1 & -2 & 1 \end{bmatrix}$$

$$\text{Thus, } D = P^{-1} B P \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 3 & 1 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Here, B is similar to D .

Theorem: An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof: Suppose A is $n \times n$ square matrix.

($\Rightarrow$): Assume that A is diagonalizable matrix. Thus, there exists a non-singular matrix Q of size n

$Q = [y_1 | y_2 | y_3 | \dots | y_n]$ and a diagonal matrix

$$E = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} = [d_1 e_1 | d_2 e_2 | \dots | d_n e_n] \text{ such that } E = Q^{-1} A Q.$$

Then consider,

$$\begin{aligned} [A y_1 | A y_2 | \dots | A y_n] &= A [y_1 | y_2 | \dots | y_n] \quad (\text{by defn matrix multiplication}) \\ &= A Q \quad (\because [y_1 | y_2 | \dots | y_n] = Q) \\ &= I A Q \quad (\text{by defn of identity matrix}) \\ &= Q Q^{-1} A Q \quad (\because I = Q^{-1} Q) \\ &= Q E \quad (\because Q^{-1} A Q = E) \\ &= Q [d_1 e_1 | \dots | d_n e_n] \quad (\because E = [d_1 e_1 | \dots | d_n e_n]) \\ &= [Q(d_1 e_1) | \dots | Q(d_n e_n)] \quad (\text{by defn of mul.}) \\ &= [d_1 Q e_1 | d_2 Q e_2 | \dots | d_n Q e_n] \\ &= [d_1 y_1 | d_2 y_2 | \dots | d_n y_n] \quad (\because Q = [y_1 | y_2 | \dots | y_n]) \end{aligned}$$

This equality of matrices allows us to conclude that the individual columns are equal vectors. i.e. $A y_i = d_i y_i$ for $1 \leq i \leq n$.

In other words, y_i is the Eigenvector of A for the Eigen Value d_i , $1 \leq i \leq n$.

Here, $y_i \neq 0$ since Q is non-singular, the set containing Q 's columns, $S = \{y_1, \dots, y_n\}$ is linearly independent set.

Conversely, Let $S = \{x_1, x_2, \dots, x_n\}$ be linearly independent set of Eigenvectors of A for the Eigen values $\lambda_1, \dots, \lambda_n$ and define $R = [x_1 | x_2 | \dots | x_n]$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = [d_1 e_1 | d_2 e_2 | d_3 e_3 | \dots | d_n e_n]$$

The columns of R are the vectors of the linearly independent set of S and so by the matrix R is non-singular. We know R^{-1} exists.

$$\begin{aligned} \text{Thus, } R^{-1}AR &= R^{-1}A[x_1 | x_2 | \dots | x_n] \\ &= R^{-1}[Ax_1 | Ax_2 | \dots | Ax_n] \quad (\text{by defn of matrix multiplication}) \\ &= R^{-1}[\lambda_1 x_1 | \lambda_2 x_2 | \dots | \lambda_n x_n] \quad (\because x_i \text{ is the Eigen Vectors of } A \text{ corresponding to } \lambda_i, \text{ so } Ax_i = \lambda_i x_i) \\ &= R^{-1}[\lambda_1 R e_1 | \dots | \lambda_n R e_n] \\ &= R^{-1}[R[d_1 e_1 | \dots | d_n e_n]] \quad (\text{by defn of matrix multiplication}) \\ &= [R^{-1}R][d_1 e_1 | \dots | d_n e_n] \\ &= I_n[d_1 e_1 | \dots | d_n e_n] \quad (\because R^{-1}R = I_n) \\ &= I_n D \quad (\because D = [d_1 e_1 | \dots | d_n e_n]) \end{aligned}$$

So, $R^{-1}AR = D$. $(\because I_n \text{ is multiplicative identity})$

This says that A is similar to diagonal matrix D via the non-singular matrix R . Thus A is diagonalizable matrix.

and hence, square matrix A is said to be diagonalizable iff it has n linearly independent eigen vectors.

Remark (1) Suppose the Eigen values $\lambda_1, \dots, \lambda_n$ are all different. then it is automatic that the eigen vectors $x_1, \dots, x_n$ are independent.

✓ Any matrix that has no repeated eigen value can be diagonalizable.

(2) We can multiply eigen values by any non-zero constants. then $Ax = \lambda x$ will remain true.

(3) An $n \times n$ real matrix is said to be diagonalizable if it is similar to real diagonal matrix. i.e. if $P^{-1}AP$ is diagonal for some invertible matrix P .

(4) Suppose that A is diagonalizable say $P^{-1}AP = D$, diagonal matrix. then $A^k = PD^kP^{-1}$, $\forall k \in \mathbb{N}$.

(5) An operator $T: V \rightarrow V$ is said to be diagonalizable if its matrix representation is diagonalizable where V is n -dimensional signal vector space.

(6) A matrix is orthogonally diagonalizable if there exists an orthogonal matrix P such that $D = P^TAP$.

(7) If no eigen values of A is repeated, then A is diagonalizable. The converse is not true.

Eg. Show that $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ is diagonalizable.

Defn: (1) A unitary matrix U is a complex square matrix that has orthonormal columns.

(2) Among complex matrices, the special class contains the Hermitian matrices: $A = A^H$. The condition on the entries is $a_{ij} = \bar{a}_{ji}$. In this case A is Hermitian.

✓ Every real symmetric matrix is Hermitian b/c its conjugate has no effect.

Minimal polynomial

Defn: Minimal polynomial of a matrix is defined as the monic polynomial of the least degree satisfied by the matrix.

✓ Minimal polynomial of a matrix A will be denoted by $M_A(x)$.

Example Find the minimal polynomial of a matrix A .

$$(1) A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} \quad (2) A = \begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(3) A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

So/ (1) Here the characteristic polynomial of A is given by $|A - \lambda I| = 0$

$$\text{So, } \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & -1 \\ 0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[4-4\lambda-\lambda+\lambda^2+2] + 0 + 0 = 0$$

$$\Rightarrow (2-\lambda)[\lambda^2-5\lambda+6] = 0 \text{ is the characteristic polynomial.}$$

$$\Rightarrow (2-\lambda)[(\lambda-2)(\lambda-3)] = 0 \Rightarrow$$

$$\Rightarrow [2\lambda^2-10\lambda+12]-\lambda^3+5\lambda^2-6\lambda = 0$$

$$\Rightarrow -\lambda^3+12\lambda^2-16\lambda+12 = 0 \Rightarrow \lambda^3-7\lambda^2+16\lambda-12 = 0 \text{ is the characteristic polynomial.}$$

$$\Rightarrow \lambda^3-7\lambda^2+16\lambda-12 = 0 \Rightarrow (\lambda-2)(\lambda^2-5\lambda+6) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow (\lambda-2)^2(\lambda-3) = 0$$

Thus, the possible minimal polynomial is $f(\lambda) = (A-2I)(A-3I)$ or $g(\lambda) = (A-2I)^2(A-3I)$

$$\text{So, } (A-2I)(A-3I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

Thus, $(A-2I)(A-3I)$ is not the minimal polynomial.

$$(A-2I)^2(A-3I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}^2 \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $(\lambda-2)^2(\lambda-3)$ is the minimal polynomial of A .

Remark! (1) the minimal polynomial of a matrix divides every polynomial satisfied by the matrix.

(2) the characteristic polynomial and minimal polynomial have the same irreducible polynomial.

(3) the scalar λ is an eigen value of a matrix A iff it is a root of the minimal $m_A(x)$.

(4) if A is $n \times n$ real matrix, then there exists a polynomial $p(\lambda)$ such that $p(A) = 0$.

(5) Minimal polynomial of A is unique.

(6) If p is some polynomial such that $p(A)=0$, then m divides p .

(7) If λ is an eigen value of A , then it is a root of m .

Cayley-Hamilton Theorem

Every square matrix A satisfies its own characteristic equation.

Proof Ex.

Ex. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Soln: Let A denotes the given matrix. Since $\det A \neq 0$ then A^{-1} exists.

→ The characteristic polynomial of A is $\lambda^3 - 4\lambda^2 + 4\lambda - 1 = (\lambda-1)^2(\lambda-2) - (\lambda-1)$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 4\lambda - 1 = (\lambda-1)^2(\lambda-2) - (\lambda-1)$$

By Cayley-Hamilton theorem, we get $A^3 - 4A^2 + 4A - I = 0$.

$$\Rightarrow A^3 - 4A^2 + 4A = I$$

$$\Rightarrow A(A^2 - 4A + 4I) = I \Rightarrow A^{-1}A(A^2 - 4A + 4I) = A^{-1}I$$

$$\Rightarrow A^2 - 4A + 4I = A^{-1}$$

$$\text{So, } A^{-1} = A^2 - 4A + 4I$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^2 - 4 \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

Remark! (1) The characteristic is not only the equation that matrix satisfies.

(2) The above theorem can be used successfully in computing the inverse of a matrix w/c is non-singular matrix.

(3) The inverse of the matrix is the polynomial of the matrix.

QED